

Relative Frobenius Formula

Avraham Aizenbud, Nir Avni and Yoav Krauz

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Abstract

For a finite group G , Frobenius found a formula for the values of the function $\sum_{\text{Irr } G} (\dim \pi)^{-s}$ for even integers s , where $\text{Irr } G$ is the set of irreducible representations of G . We generalize this formula to the relative case: for a subgroup H , we find a formula for the values of the function $\sum_{\text{Irr } G} (\dim \pi)^{-s} (\dim \pi^H)^{-t}$. We apply our results to compute the E-polynomials of Fock–Goncharov spaces and to relate the Gelfand property to the geometry of generalized Fock–Goncharov spaces.

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1 Frobenius' formula

Let S be a compact surface and let G be a finite group. A fundamental formula of Frobenius relates the number of homomorphisms from the fundamental group of S to G and the dimensions of the irreducible representations of G :

Theorem 1.1. *Let S be a compact surface of genus k and let G be a finite group. Then,*

$$|G|^{2k-1} \sum_{\pi \in \text{Irr } G} (\dim \pi)^{2-2k} = |\text{Hom}(\pi_1(S), G)| = |\{(x_1, y_1, \dots, x_k, y_k) \in G^{2k} \mid [x_1, y_1] \cdots [x_k, y_k] = 1\}|,$$

where $\text{Irr } G$ is the set of (isomorphism classes of) irreducible representations of G .

For example, $k = 0$ gives $\sum_{\pi \in \text{Irr } G} (\dim \pi)^2 = |G|$, whereas from $k = 1$ we get

$$|\text{Irr } G| = \frac{1}{|G|} \cdot |\{(x, y) \in G^2 \mid xy = yx\}| = \sum_{x \in G} \frac{|C_G(x)|}{|G|} = \sum_{x \in G} \frac{1}{|x^G|} = |G//G|.$$

Theorem 1.1 also has versions for compact Lie groups and for pro-finite groups (see [Wit91, AA]).

Theorem 1.1 is the case $g = 1$ of the following theorem:

Theorem 1.2. *Let G be a finite group and let $g \in G$. Then,*

$$|G|^{2k-1} \sum_{\pi \in \text{Irr } G} (\dim \pi)^{1-2k} \chi_\pi(g) = |\{(x_1, y_1, \dots, x_k, y_k) \in G^{2k} \mid [x_1, y_1] \cdots [x_k, y_k] = g\}|.$$

In this paper, we generalize Frobenius' formula to the relative case, i.e., we replace the representation theory of a group G by the harmonic analysis on some G -space X . We apply our result for Gelfand pairs and the Hodge theory of Fock–Goncharov spaces.

2 Relative representation theory

Relative representation theory is motivated by the following example:

Example 2.1. *Let H be a (finite) group, and consider H as a $H \times H$ -set via the action*

$$(h_1, h_2) \cdot h := h_1 h h_2^{-1}.$$

Consider the space $\mathbb{C}[H]$ of complex-valued functions on H as a representation of $H \times H$. We have

$$\mathbb{C}[H] = \bigoplus_{\pi \in \text{Irr } H} \pi \otimes \pi^*.$$

This example shows that understanding the $H \times H$ -representation $\mathbb{C}[H]$ “is the same” as understanding the representation theory of H . One can reformulate many concepts of the representation theory of H in terms of the $H \times H$ -representation $\mathbb{C}[H]$. Relative representation theory (also known as abstract harmonic analysis) deals with those concepts considered in a wider generality: a group G acting on a set X and the representation of G on $\mathbb{C}[X]$.

Two important examples of representation theoretical concepts that have relative counterparts are Schur’s Lemma, whose relative counterpart is the Gelfand property (see Definition 4.1 below) and the notion of a character, whose relative counterpart is the notion of spherical (or relative) character (see Definition B.1 below).

3 Relative version of Frobenius’ formula

We prove the following theorem in §6:

Theorem 3.1. *Let G be a finite group acting on a finite set X , let $g \in G$, and let $k \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{\geq 1}$. Then:*

$$\begin{aligned} \sum_{\pi \in \text{irr} G} \frac{\dim(\text{Hom}_G(\pi, \mathbb{C}[X]))^m}{\dim \pi^{m+2k-1}} \chi_\pi(g) &= \frac{1}{\#G^{m+2k-1}} \cdot \\ &\cdot \#\{p_1, \dots, p_m \in X, h_1, \dots, h_m, a_1, \dots, a_k, b_1, \dots, b_k \in G \mid h_i \in G_{p_i}, \prod_{i=1}^m h_i \cdot \prod_{i=1}^k [a_i, b_i] = g\} = \\ &= \frac{1}{\#G^{m+2k-1}} \sum_{h_2, \dots, h_m, a_1, \dots, a_k, b_1, \dots, b_k \in G} \#X^{g^{-1} \cdot h_2 \cdots h_m \cdot [a_1, b_1] \cdots [a_k, b_k]} \prod_{i=2}^m \#X^{h_i}, \end{aligned}$$

where $[a, b] := aba^{-1}b^{-1}$ is the commutator of a and b .

In Appendix B we reformulate this theorem in terms of spherical characters.

4 A criterion for Gelfand pairs

Recall the definition of Gelfand pairs:

Definition 4.1. *Let G be a finite group.*

1. *Assume that G acts on a finite set X . We say that X is multiplicity free if, for any $\pi \in \text{Irr}(G)$, we have $\dim \text{Hom}_G(\pi, \mathbb{C}[X]) \leq 1$.*

2. Let $H < G$. We say that (G, H) is a Gelfand pair if G/H is a multiplicity free G -set.

Theorem 3.1 gives us the following criterion for Gelfand pairs:

Corollary 4.2. *Let $H \subset G$ be a pair of groups, and let $X = G/H$. Then the pair (G, H) is a Gelfand pair if and only if*

$$\sum_{g,h \in G} \#X^{[g,h]} = \sum_{g,h \in G} \#X^g \cdot \#X^h \cdot \#X^{gh}.$$

In fact, Theorem 3.1 implies also the following more general statement:

Corollary 4.3. *Let $H \subset G$ be a pair of groups and let $X = G/H$. For every $k, m \in \mathbb{Z}_{\geq 0}$ denote:*

$$f(k, m) := \sum_{h_1, \dots, h_m, a_1, \dots, a_k, b_1, \dots, b_k \in G} \#X^{h_1 \cdots h_m \cdot [a_1, b_1] \cdots [a_k, b_k]} \prod_{i=1}^m \#X^{h_i}.$$

Then, the following are equivalent:

- *The pair (G, H) is a Gelfand pair.*
- *For every $k, m \in \mathbb{Z}_{\geq 0}$ and $0 < l \leq k$, we have $f(k-l, m) = f(k, m+2l)$.*
- *For some $k, m \in \mathbb{Z}_{\geq 0}$ and $0 < l \leq k$, we have $f(k-l, m) = f(k, m+2l)$.*

5 Fock–Goncharov spaces

Theorem 3.1 can also be interpreted as a counting formula for (generalized) Fock–Goncharov spaces, which we proceed to define. The setting for this section is as follows: let \overline{S} be a compact surface, let $p_1, \dots, p_m \in \overline{S}$, $m \geq 1$, be distinct points, and denote $S = \overline{S} \setminus \{p_1, \dots, p_m\}$. Such S is called a surface of finite type. Choose a base point $s \in S$ and, for each $i = 1, \dots, m$, choose a representative $\tau_i \in \pi_1(S, s)$ from the conjugacy class corresponding to a circle around p_i .

Definition 5.1. *Let G be a group acting on a set X . An X -framed representation $\pi_1(S, s) \rightarrow G$ is a tuple (ρ, x_1, \dots, x_m) , where $\rho : \pi_1(S, s) \rightarrow G$ is a homomorphism, and $x_i \in X$ satisfy $\rho(\tau_i)x_i = x_i$. The collection of all X -framed representations is denoted by $\hat{\mathcal{X}}_{S, s, (\tau_i), G, X}$.*

If s' and τ'_i are different choices of a point and loops, then there is a bijection (depending on a choice of a path from s to s') between $\widehat{\mathcal{X}}_{S,s,(\tau_i),G,X}$ and $\widehat{\mathcal{X}}_{S,s',(\tau'_i),G,X}$. When no confusion arises, we will omit s and τ_i from the notation.

If \mathbf{G} is group scheme acting on a scheme \mathbf{X} , then the functor sending a scheme T to $\widehat{\mathcal{X}}_{S,\mathbf{G}(T),\mathbf{X}(T)}$ is representable by a scheme that we denote by $\widehat{\mathcal{X}}_{S,\mathbf{G},\mathbf{X}}$.

Definition 5.2. Let \mathbf{G} be a group scheme acting on a scheme \mathbf{X} . Then, \mathbf{G} acts on $\widehat{\mathcal{X}}_{S,\mathbf{G},\mathbf{X}}$, and we denote the quotient stack by $\mathcal{X}_{S,\mathbf{G},\mathbf{X}}$. Similarly, if a group G acts on a set X , we denote the quotient groupoid $G \backslash \widehat{\mathcal{X}}_{S,G,X}$ by $\mathcal{X}_{S,G,X}$.

Remark 5.3.

- If \mathbf{X} is the flag variety of a reductive group \mathbf{G} , then the stack $\mathcal{X}_{S,\mathbf{G},\mathbf{X}}$ was defined in [FG06]. The authors of [FG06] defined the notion of a framed \mathbf{G} local system and showed that $\mathcal{X}_{S,\mathbf{G},\mathbf{X}}$ is the moduli stack of framed \mathbf{G} local systems on S (see [FG06, §2]). The notion of a framed \mathbf{G} local system extends to general \mathbf{G} and \mathbf{X} , and the same proof shows that $\mathcal{X}_{S,\mathbf{G},\mathbf{X}}$ is the moduli space of framed (\mathbf{G}, \mathbf{X}) -local systems.
- If \mathbf{G} is connected, then, by Lang's Theorem, $\mathcal{X}_{S,\mathbf{G},\mathbf{X}}(\mathbb{F}_p) \cong \mathcal{X}_{S,\mathbf{G}(\mathbb{F}_p),\mathbf{X}(\mathbb{F}_p)}$.

In terms of the definitions above, Theorem 3.1 implies:

Theorem 5.4. Let G be a finite group acting on a finite set X . Then

$$\#\widehat{\mathcal{X}}_{S,G,X} = (\#G)^{1-\chi(S)} \sum_{\pi \in \text{Irr } G} \frac{\dim(\text{Hom}_G(\pi, \mathbb{C}[X]))^{\#(\overline{S} \setminus S)}}{\dim \pi^{-\chi(S)}},$$

and

$$\text{vol}(\mathcal{X}_{S,G,X}) := \sum_{x \text{ is an isomorphism class of } \mathcal{X}_{S,G,X}} \frac{1}{\# \text{Aut}(x)} = (\#G)^{-\chi(S)} \sum_{\pi \in \text{Irr } G} \frac{\dim(\text{Hom}_G(\pi, \mathbb{C}[X]))^{\#(\overline{S} \setminus S)}}{\dim \pi^{-\chi(S)}}.$$

Corollary 5.5. Let G be a finite group acting on a finite set X . The following are equivalent:

- X is a multiplicity free G -space.
- For any two non-compact surfaces of finite type S_1, S_2 such that $\chi(S_1) = \chi(S_2)$, we have $\text{vol}(\mathcal{X}_{S_1,G,X}) = \text{vol}(\mathcal{X}_{S_2,G,X})$.
- There are two non homeomorphic non-compact surfaces of finite type S_1, S_2 such that $\chi(S_1) = \chi(S_2)$ and $\text{vol}(\mathcal{X}_{S_1,G,X}) = \text{vol}(\mathcal{X}_{S_2,G,X})$.

Definition 5.6. We say that a set T of prime powers is dense if, for any finite Galois extension E/\mathbb{Q} and for any conjugacy class $\gamma \subset \text{Gal}(E/\mathbb{Q})$, there exists $p^n \in T$ such that p is unramified in E and $\gamma = \text{Fr}_p^n$.

Remark 5.7.

- The Chebotarev Density Theorem says that the set of all primes is dense.
- The Grothendieck trace formula implies that if X_1, X_2 are two schemes such that $X_1(\mathbb{F}_q) = X_2(\mathbb{F}_q)$ when q ranges over a dense set of prime powers, then $X_1(\mathbb{F}_{p^n}) = X_2(\mathbb{F}_{p^n})$ for almost all primes p and for all natural numbers n .

The last corollary and [Kat08] implies:

Corollary 5.8. Let \mathbf{G} be a group scheme over \mathbb{Z} acting on a scheme \mathbf{X} . The following are equivalent:

- There is a dense set T of prime powers such that, for any $q \in T$, the set $\mathbf{X}(\mathbb{F}_q)$ is a multiplicity free $\mathbf{G}(\mathbb{F}_q)$ space.
- For all but finitely many primes p and for all n , the set $\mathbf{X}(\mathbb{F}_{p^n})$ is a multiplicity free $\mathbf{G}(\mathbb{F}_{p^n})$ space.

Moreover, if these conditions hold then, for any two non-compact surfaces S_1, S_2 such that $\chi(S_1) = \chi(S_2)$, the varieties $\hat{\mathcal{X}}_{S_1, \mathbf{G}, \mathbf{X}}$ and $\hat{\mathcal{X}}_{S_2, \mathbf{G}, \mathbf{X}}$ have the same E -polynomial¹.

We will now apply Theorem 5.4 for the case of \mathbf{GL}_n acting on its flag variety \mathbf{Fl}_n . Recall that, if $\lambda = (\lambda_1, \dots, \lambda_m)$ is a partition of n and λ^* is the conjugate partition, then

$$h_\lambda(i, j) = \lambda_i - j + \lambda_j^* - i + 1$$

is the length of the hook in the Young diagram corresponding to λ passing through the box (i, j) . We prove the following:

Theorem 5.9.

•

$$\text{vol}(\mathcal{X}_{S, \mathbf{GL}_n, \mathbf{Fl}_n}(\mathbb{F}_q)) = (n!)^{\#\bar{S} \setminus S} \sum_{\lambda \text{ is a partition of } n} q^{\sum_k (k-1)\lambda_k \chi(S)} \prod_{i, j: j \leq \lambda_i} \frac{(q^{h_\lambda(i, j)} - 1)^{-\chi(S)}}{h_\lambda(i, j)^{\#\bar{S} \setminus S}}.$$

¹For the definition of the E-polynomial see e.g. [Kat08]

- The E polynomial of $\widehat{\mathcal{X}}_{S, \mathbf{GL}_n, \mathbf{Fl}_n}$ is

$$(n!)^{\#\overline{S} \setminus S} \prod_{k=1}^n (x^n y^n - x^k y^k) \sum_{\lambda} (xy)^{\sum_k (k-1) \lambda_k \chi(S)} \prod_{i,j: j \leq \lambda_i} \frac{((xy)^{h_{\lambda}(i,j)} - 1)^{-\chi(S)}}{h_{\lambda}(i,j)^{\#\overline{S} \setminus S}}.$$

For the proof, we collect the following facts:

Proposition 5.10 ([Jam84]). *For every partition λ of n , there exists a unique irreducible representation R_{λ} of $\mathbf{GL}_n(\mathbb{F}_q)$ satisfying:*

- R_{λ} appears in the permutation representation $\mathbb{C}[\mathbf{GL}_n(\mathbb{F}_q)/\mathbf{P}_{\lambda}(\mathbb{F}_q)]$, where \mathbf{P}_{λ} is the standard parabolic corresponding to λ (see [Jam84, Chapter 11]).
- R_{λ} does not appear in the permutation representation $\mathbb{C}[\mathbf{GL}_n(\mathbb{F}_q)/\mathbf{P}_{\mu}(\mathbb{F}_q)]$, for $\mu < \lambda$ (see [Jam84, Chapter 15]).

•

$$\dim R_{\lambda} = q^{\sum_k (k-1) \lambda_k} \frac{\# \mathbf{GL}_n(\mathbb{F}_q)}{\prod_{i,j: j \leq \lambda_i} (q^{h_{\lambda}(i,j)} - 1)}.$$

(see [Jam84, Page 2]).

Let $\mathbf{B} \subset \mathbf{GL}_n$ be the standard Borel. Taking $T_{\lambda} = R_{\lambda}^{\mathbf{B}(\mathbb{F}_q)}$, we get

Corollary 5.11. *For every partition λ of n , we have*

- T_{λ} appears in the representation $\mathbb{C}[\mathbf{GL}_n(\mathbb{F}_q)/\mathbf{P}_{\lambda}(\mathbb{F}_q)]^{\mathbf{B}(\mathbb{F}_q)}$.
- T_{λ} does not appear in the representation $\mathbb{C}[\mathbf{GL}_n(\mathbb{F}_q)/\mathbf{P}_{\mu}(\mathbb{F}_q)]^{\mathbf{B}(\mathbb{F}_q)}$, for $\mu < \lambda$.

The following is classical:

Proposition 5.12. *For every partition λ of n , there exists a unique irreducible representation π_{λ} of S_n satisfying:*

- π_{λ} appears in the permutation representation $\mathbb{C}[S_n/S_{\lambda}]$, where $S_{(\lambda_1, \dots, \lambda_m)} = S_{\lambda_1} \times \dots \times S_{\lambda_m} \subset S_n$.
- π_{λ} does not appear in the permutation representation $\mathbb{C}[S_n/S_{\mu}]$, for $\mu < \lambda$.
- $\dim \pi_{\lambda} = \frac{n!}{\prod_{i,j: i \leq \lambda_j} h_{\lambda}(i,j)}.$

Proof of Theorem 5.9. Since $\dim \text{Hom}(R_\lambda, \mathbb{C}[\mathbf{F}_n]) = \dim T_\lambda$, it is enough to show that $\dim T_\lambda = \dim \pi_\lambda$, for every λ . Recall that the Hecke algebra $H^{S_n}(t)$ corresponding to the Coxeter group S_n is a (polynomial) one parameter family of algebras whose underlying vector space is $\mathbb{C}[S_n]$; we denote the product in $H^{S_n}(t)$ by $*_t$. Recall that the product $*_1$ is the convolution on $\mathbb{C}[S_n]$ and that, if t is a prime power, then the product $*_t$ corresponds to the convolution in $\mathbb{C}[\mathbf{B}(\mathbb{F}_t) \backslash \mathbf{GL}_n(\mathbb{F}_t) / \mathbf{B}(\mathbb{F}_t)]$ under the identification $\mathbb{C}[\mathbf{B}(\mathbb{F}_t) \backslash \mathbf{GL}_n(\mathbb{F}_t) / \mathbf{B}(\mathbb{F}_t)] \cong \mathbb{C}[S_n]$ given by the Bruhat decomposition. Let $M_\lambda(t) \subset H^{S_n}(t)$ be the subspace of S_λ -(right)-invariant elements of $\mathbb{C}[S_n]$. For every prime power t , $M_\lambda(t)$ is an ideal, and, hence, the same is true for every t . Using the interpolation of the natural inner product, we get that, for $t \in \mathbb{R}_{\geq 1}$, the algebra $H^{S_n}(t)$ is semisimple, and, hence, there is an (analytic) trivialization of $H^{S_n}(t)$ over $\mathbb{R}_{\geq 1}$. Since there are only finitely many isomorphism types of representations of a given dimension, we get that $M_\lambda(t)$ can also be trivialized over $\mathbb{R}_{\geq 1}$. Corollary 5.11 and Proposition 5.12 imply that, under the algebra isomorphism $\mathbb{C}[S_n] \rightarrow \mathbb{C}[\mathbf{B}(\mathbb{F}_q) \backslash \mathbf{GL}_n(\mathbb{F}_q) / \mathbf{B}(\mathbb{F}_q)]$, the modules T_λ and π_λ are isomorphic, and hence have the same dimension. \square

6 Proof of Theorem 3.1

The case $k = 0$, $m = 1$ of theorem 3.1 is easy:

Lemma 6.1. *Let G be a finite group acting on a finite set X . Then:*

$$\sum_{\pi \in \text{Irr } G} \dim(\text{Hom}_G(\pi, \mathbb{C}[X])) \cdot \chi_\pi(g) = \chi_{\mathbb{C}[X]}(g) = \#X^g. \quad (1)$$

In order to deduce the general case we need a basic fact about convolution of characters. Recall that for two functions $f, g \in \mathbb{C}[G]$, the convolution is defined by

$$(f * g)(h) = \sum_{u \in G} f(u)g(u^{-1}h).$$

Lemma 6.2. *For any $\pi, \tau \in \text{Irr } G$ we have:*

$$\chi_\pi * \chi_\tau = \frac{\delta_{\pi, \tau} \#G}{\dim(\pi)} \chi_\pi.$$

Now we ready to prove the main theorem.

Proof of theorem 3.1. Applying Lemma 6.2, the assertion follows by convolving (1) with itself m times and with the formula in Theorem 1.2. \square

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A An alternative proof of the Frobenius formula

Lemma 6.1 gives an alternative proof of the Frobenius formula (Theorem 1.1).

Let G be a finite group acting on a finite set X . For a representation π of G , define a function on $X \times X$ by

$$\chi_\pi^X(x, y) = \frac{1}{\#G} \sum_{h: hx=y} \chi_\pi(h). \quad (2)$$

Lemma A.1. *Consider the 2-sided action of $G \times G$ on G . Let π be a representation of G . Then*

$$\chi_{\pi \otimes \pi^*}^G(1, g) = \frac{1}{\#G \dim \pi} \chi_\pi(g).$$

Proof.

$$\chi_{\pi \otimes \pi^*}^G(1, g) = \frac{1}{\#G} \sum_{h_1, h_2: h_1 h_2^{-1} = g} \chi_\pi(h_1) \chi_\pi(h_1^{-1}) = \frac{(\chi_\pi * \chi_\pi)(g)}{\#G} = \frac{1}{\#G \dim \pi} \chi_\pi(g),$$

where the last equality is by Lemma 6.2 □

Proof of Theorem 1.1. the case $k = 1$ follows from the Lemma 6.1 and lemma A.1. The general case follows by taking convolution power of the case $k = 1$ and using Lemma 6.2. □

B The spherical character

The relative counterpart of the notion of the character of a representation is given in the following definition:

Definition B.1. *Let G be a finite group acting on a finite set X . Let π be a representation of G .*

1. Let $\phi : \pi \rightarrow \mathbb{C}[X]$ and $\psi : \pi^* \rightarrow \mathbb{C}[X]$ be morphisms of representations. Denote by ϕ^t and ψ^t the dual maps. We define the spherical character $\chi_\pi^{\phi \otimes \psi} \in \mathbb{C}[X \times X]$ by

$$\chi_\pi^{\phi \otimes \psi}(x, y) = \langle \phi^t(\delta_x), \psi^t(\delta_y) \rangle,$$

where $\delta_x \in \mathbb{C}[X] = \mathbb{C}[X]^*$ is the Kronecker delta function supported at x .

2. This definition extends (by linearity) to the case when $\phi \otimes \psi$ is replaced by any element of $\text{Hom}(\pi, \mathbb{C}[X]) \otimes \text{Hom}(\pi^*, \mathbb{C}[X]) = \text{End}(\text{Hom}(\pi, \mathbb{C}[X]))$.

Lemma B.2.

$$\chi_\pi^X := \chi_\pi^{Id_{\text{Hom}(\pi, \mathbb{C}[X])}}.$$

Proof. For $x \in X$, let $L_\pi^x : \text{Hom}_G(\pi, \mathbb{C}[X]) \rightarrow \pi^*$ be the linear map defined by

$$\phi \in \text{Hom}_G(\pi, \mathbb{C}[X]) \mapsto (u \in \pi \mapsto \phi(u)(x)).$$

Note that $\text{Hom}_G(\pi, \mathbb{C}[X])$, $\text{Hom}_G(\pi^*, \mathbb{C}[X])$ are naturally dual to each other by the pairing

$$\langle \phi, \psi \rangle := \sum_{x \in X} \langle L_\pi^x \phi, L_{\pi^*}^x \psi \rangle \quad (\phi \in \text{Hom}_G(\pi, \mathbb{C}[X]); \psi \in \text{Hom}_G(\pi^*, \mathbb{C}[X]))$$

therefore we shall identify $\text{Hom}_G(\pi^*, \mathbb{C}[X])$ with $\text{Hom}_G(\pi, \mathbb{C}[X])^*$.

Let $\phi \in \text{Hom}_G(\pi, \mathbb{C}[X])$, $\psi \in \text{Hom}_G(\pi^*, \mathbb{C}[X])$. Then by definition,

$$\chi_\pi^{\phi \otimes \psi}(x, y) = \langle \phi^t(\delta_x), \psi^t(\delta_y) \rangle = \langle L_\pi^x \phi, L_{\pi^*}^y \psi \rangle = \langle (L_{\pi^*}^y)^t L_\pi^x \phi, \psi \rangle,$$

so $\chi_\pi^{Id_{\text{Hom}(\pi, \mathbb{C}[X])}}(x, y) = \text{tr}((L_{\pi^*}^y)^t L_\pi^x)$.

It is easy to see that $(L_\pi^x)^t : \pi \rightarrow \text{Hom}_G(\pi^*, \mathbb{C}[X])$ can be computed by

$$\forall u \in \pi, f \in \pi^* : ((L_\pi^x)^t u)(f) = \frac{1}{\#G} \sum_{h \in G} f(\pi(h)u) \delta_{hx}.$$

Now, $\chi_\pi^{Id_{\text{Hom}(\pi, \mathbb{C}[X])}} = \text{tr}((L_{\pi^*}^y)^t L_\pi^x) = \text{tr}(L_\pi^x (L_{\pi^*}^y)^t)$. Note that $L_\pi^x (L_{\pi^*}^y)^t$ is the linear mapping $\pi^* \rightarrow \pi^*$ defined by

$$\forall f \in \pi^* : (L_\pi^x (L_{\pi^*}^y)^t) f = \left(u \in \pi \mapsto \frac{1}{\#G} \sum_{h \in G} \langle u, (\pi^*(h)f) \rangle \delta_{hy, x} \right) = \frac{1}{\#G} \sum_{h \text{ s.t. } hy=x} \pi^*(h)f$$

so

$$\chi_\pi^{Id_{\text{Hom}(\pi, \mathbb{C}[X])}} = \text{tr}(L_\pi^x (L_{\pi^*}^y)^t) = \frac{1}{\#G} \sum_{h \text{ s.t. } hy=x} \chi_{\pi^*}(h) = \frac{1}{\#G} \sum_{h \text{ s.t. } hx=y} \chi_\pi(h) = \chi_\pi^X(x, y)$$

□

We reformulate Theorem 3.1 in terms of the spherical character:

Theorem B.3. *Let G be a finite group that acts on a finite set X . Then:*

$$\sum_{\pi \in \text{Irr } G} \frac{\dim(\text{Hom}_G(\pi, \mathbb{C}[X]))^m}{\dim \pi^{m+2k-1}} \chi_\pi^X(x_1, x_2) = \frac{1}{\#G^{m+2k}} \cdot \#\{p_1, \dots, p_m \in X, h_1, \dots, h_m, a_1, \dots, a_k, b_1, \dots, b_k \in G \mid h_i \in G_{p_i}, \prod_{i=1}^m h_i \cdot \prod_{i=1}^k [a_i, b_i] \cdot x_1 = x_2\}.$$

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